UNIVERSITY OF CALIFORNIA, BERKELEY

Computing Jordan Form Using Jordan Chain

EE221A Linear System Theory

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1 JORDAN CHAIN

Definition 1 (Jordan Chain)**.** A Jordan chain of length μ associated with eigenvalue $\lambda \in \mathbb{C}$ of a matrix $A \in \mathbb{R}^{n\times n}$ is a sequence of vectors* $\{v^j\}_{j=1}^{\mu} \subseteq \mathbb{C}^n$ such that the following conditions hold:

1. Elements in $\{v^j\}_{j=1}^{\mu}$ are linearly independent;
2. $(A-\lambda I)v^1 = 0$, where v^1 is the normal eigenvector associated with λ;
3. $(A-\lambda I)v^j = v^{j-1}$, where $v^j,\ j>1$ is the generalized eigenvector.

Definition 2 (Maximal Jordan Chain)**.** A Jordan chain $\{v^j\}_{j=1}^{\mu}$ is maximal if it cannot be extended using Definition.1, i.e. there does not exist a generalized eigenvector $v \in \mathbb{C}^n$ linearly independent from $\{v^j\}_{j=1}^{\mu}$ such that $(A-\lambda I)v = v^{\mu}$.

Fact 1 (Necessary condition of generalized eigenvectors)**.** For the jth (generalized) eigenvector v^j in the Jordan chain $\{v^j\}_{j=1}^{\mu}$:

$$v^j \in \{v^j\}_{j=1}^{\mu} \to v^j \in \mathcal{N}(A-\lambda I)^j \Leftrightarrow (A-\lambda I)^j v^j = 0$$

Remark. We should NEVER use Fact.1 to compute Jordan chain or Jordan form.

Fact 2. $v^j \notin \mathcal{N}(A-\lambda I)^l,\ l = 1,2,\dots,j-1$

Fact 3. $\mathcal{N}(A-\lambda I)^j \subseteq \mathcal{N}(A-\lambda I)^{j+1}$

Fact 4. Let the minimal polynomial of matrix $A \in \mathbb{R}^{n\times n}$ to be $\hat{\psi}_A(s) = (s-\lambda_1)^{m_1}\dots(s-\lambda_\sigma)^{m_\sigma}$. Let $\{v_i^j\}_{j=1}^{\mu_i}$ be the *longest* Maximal Jordan chain associated with λ_i

$$m_i = \mu_i = \text{size of the largest Jordan Block associated with } \lambda_i$$

* j is the index of eigenvectors in the Jordan chain, not the power.

2 Computing Jordan From Using Jordan Chain

Given a matrix $A \in \mathbb{R}^{n\times n}$, we want to compute its Jordan form $A = T^{-1}JT$.

Step 1 Obtain the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_\sigma\}$ and associated eigenvectors $\{e_1, e_2, \dots, e_k\}$. Note: $\sigma \le k$.

Step 2 First to compute the Jordan chains for λ_1. Assume λ_1 has two eigenvectors $\{e_1, e_2\}$. For each e_i, construct its Maximal Jordan chain using $(A - \lambda_1 I)v_i^j = v_i^{j-1}$, $j > 1$ (Definition.1). We will end up with two Jordan chains like $\{e_1, v_1^2, \dots\}$ and $\{e_2, v_2^2, \dots\}$.

Remark. Make sure v_i^j is linearly independent from *any other* normal/generalized eigenvectors computed previously (even if they are in other Jordan chains, because we want to construct a full rank matrix T).

Step 3 Repeat Step 2 for all the remaining eigenvalues. For simplicity, we assume A has two eigenvalues λ_1 and λ_2. λ_1 has normal eigenvectors e_1, e_2 and Jordan chains $\{e_1, v_1^2, v_1^3\}$, $\{e_2, v_2^2\}$. λ_2 has a normal eigenvector e_3 and the Jordan chain $\{e_3\}$. The matrices in the Jordan form $A = T^{-1}JT$ are represented by:

$$T^{-1} = [e_1 \; v_1^2 \; v_1^3 \; e_2 \; v_2^2 \; e_3]$$

$$J = \begin{bmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & 1 & & & \\ & & \lambda_1 & & & \\ & & & \lambda_1 & 1 & \\ & & & & \lambda_1 & \\ & & & & & \lambda_2 \end{bmatrix}$$

Exercise 1. Try to relate Jordan chain with EE221A Lecture Notes 13 page 8.

3 Examples

Example 1 (EE221A Discussion 9 Problem 4). *Consider the following matrix:*

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Find its Jordan form $A = T^{-1}JT$.

Solution First observe that the eigenvalues of matrix A are $\lambda = 3, 3, 3$.
Now compute the normal eigenvectors e:

$$(A - \lambda I)e = 0 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} e = 0 \rightarrow e = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix}$$

This implies λ has two linearly independent (normal) eigenvectors. Without loss of generality, choose them to be:

$$e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Start to construct Jordan chain for e_1:

$$(A - \lambda I)v_1^2 = e_1 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_1^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Unfortunately since $e_1 \notin \mathscr{R}(A - \lambda I)$, we cannot further extend this Jordan chain. Therefore we obtained the Maximal Jordan chain for e_1: $\{e_1\}$.
Proceed to construct the Jordan chain for e_2:

$$(A - \lambda I)v_2^2 = e_2 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow v_2^2 = \begin{bmatrix} 1 \\ b \\ c \end{bmatrix} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

At this point we have already got three linearly independent eigenvectors for constructing T^{-1} matrix. So we are confident to say that the Jordan chain for e_2 is terminated, which is $\{e_2, v_2^2\}$.
Finally, the matrices T^{-1} and J are:

$$T^{-1} = [e_1 \; e_2 \; v_2^2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 2 (EE221A Discussion 9 Problem 8). *Consider the following matrix:*

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find its Jordan form $A_3 = T^{-1}JT$.

Solution The eigenvalues of matrix A_3 are $\lambda = 0, 0, 0, 0$.
Compute the normal eigenvectors e:

$$(A_3 - \lambda I)e = 0 \rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e = 0 \rightarrow e = \begin{bmatrix} 0 \\ b \\ c \\ 0 \end{bmatrix}$$

This implies λ has two linearly independent (normal) eigenvectors. Without loss of generality, choose them to be:

$$e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Start to construct Jordan chain for e_1:

$$(A_3 - \lambda I)v_1^2 = e_1 \rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} v_1^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow v_1^2 = \begin{bmatrix} 1 \\ b \\ c \\ 0 \end{bmatrix} := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now try to extend the Jordan chain e_1, v_1^2 and get v_1^3 based on v_1^2:

$$(A_3 - \lambda I)v_1^3 = v_1^2 \rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} v_1^3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow v_1^3 = \begin{bmatrix} 0 \\ b \\ c \\ 1 \end{bmatrix} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Now our Jordan chain is $\{e_1, v_1^2, v_1^3\}$. Remember that we have the other normal eigenvector e_2. Thus we already have four linearly independent normal/generalized eigenvectors that are enough for making up T^{-1} matrix. However, we can try to continue our extension for the Jordan chain and see how it terminates:

$$(A_3 - \lambda I)v_1^4 = v_1^3 \rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} v_1^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

But $v_1^3 \notin \mathscr{R}(A_3 - \lambda I)$, which implies that the Jordan chain reaches the maximal length.
Similarly, since $e_2 \notin \mathscr{R}(A_3 - \lambda I)$, the second Jordan chain of λ is $\{e_2\}$.
Finally, the matrices T^{-1} and J are:

$$T^{-1} = [e_1\ v_2^2\ v_2^3\ e_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

† *Author: Haimin Hu, ShanghaiTech University*