

# Min-max Differential Inequalities for Polytopic Tube MPC

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**Abstract**—This paper is concerned with robust, tube-based MPC for control systems with bounded time-varying disturbances. In tube MPC, predicted trajectories are replaced by a robust forward invariant tube (RFIT), a set-valued function enclosing all possible state trajectories under a given feedback control law, regardless of the uncertainty realization. In this paper, the main idea is to characterize RFITs with polytopic cross-sections via a min-max differential inequality for their support functions. This result leads to a conservative but tractable polytopic tube MPC formulation, which can be solved using existing optimal control solvers. The corresponding theoretical developments are illustrated by a numerical case study.

## I. INTRODUCTION

Robust model predictive control (MPC) [1], [2], [3] solves at every sampling time an optimal control problem, whose decision variables are future control policies, that is, the feedback control functions, which guarantee constraint satisfaction in the presence of model uncertainties or external disturbances. In recent decades, numerous formulations leading to tractable approximations of the robust MPC problem have appeared [4], [5], [6]. Inspired by viability theory [7], [8], [9] and set-theoretic methods for control [10], [11], robust MPC formulations based on set-propagation methods have emerged as a promising methodology to tackle the synthesis of robust MPC controllers [3], [12]. The following review focuses on tube MPC methods. A more complete review of other methods can be found in [13].

Robust MPC based on parametric set-propagation, also known under the name “Tube MPC”, predict and optimize set-valued trajectories, called robust forward invariant tubes [14], [15], [3], [13]. In this context, the feedback laws, which generate the tubes can be considered as optimization variables. State-of-the-art implementations of Tube MPC are often based on a parameterization of the feedback control law [16]. In early approaches [17]—as well as modern real-time variants [18]—of robust MPC, these feedback laws are precomputed offline.

Methods that simultaneously parameterize and optimize both, the tubes as well as the control policies, can be found in [3], [19]. However, an even more recent trend in Tube MPC goes towards direct propagation of robust forward invariant tubes. This can be achieved by characterizing robust forward invariant tubes via the so-called min-max differential inequalities as introduced in [12]. Here, the main idea is

to neither store nor parameterize the control law directly such that the accuracy of the Tube MPC controller depends on the set parameterization only. The boundary control law is then never stored explicitly, although one could, at least in principle, recover this control law in the form of an parametric solution of a min-max optimization problem. The corresponding technical details of this approach can be found in [12].

Vertex control is another set-based approach for control synthesis. This technique constructs a linear state feedback controller in terms of the control inputs at the vertices of a polyhedral state set. These methods find their origins in Gutman’s work [20]. Recent advances based on an interpolating scheme between global vertex control and local unconstrained robust optimal control can be found in [21], [22]. For time-varying uncertain systems, we refer readers to [23], which deals with worst case minimal reaching time.

This work extends the developments in [12] by proposing a Tube MPC scheme based on tubes with polytopic cross-sections. Here, we introduce a min-max differential inequality, providing sufficient conditions for a time-dependent convex set-valued function to be an RFIT for a given system. This min-max differential inequality is then used to derive practical implementations of Tube MPC. The proposed scheme can be interpreted as a generalization of Gutman’s vertex controller [20].

*Outline:* The remainder of the paper is organized as follows: after introducing the problem formulation in Section II, Section III reviews the theoretical framework based on min-max differential inequalities for characterizing RFITs. Section IV presents a polytopic parameterization of RFITs. Section V presents a practical implementation of tube-based MPC and Section VI presents a case study.

*Notation:* The set of  $n$ -dimensional Lebesgue integrable functions is denoted by  $L_1^n$ . Its associated Sobolev space of weakly differentiable functions with  $L_1^n$  derivatives is denoted by  $W_{1,1}^n$ . We denote by  $\mathbb{K}^n$  and  $\mathbb{K}_C^n$  the sets of compact and convex compact sets in  $\mathbb{R}^n$ , respectively. Let  $Z \in \mathbb{K}_C^n$  be a given compact and convex set. We use the notation  $V[Z] : \mathbb{R}^n \rightarrow \mathbb{R}$  to denote the support function of  $Z$ ,

$$\forall c \in \mathbb{R}^n, \quad V[Z](c) = \max_{z \in Z} c^\top z.$$

Let  $p = [p_1, \dots, p_N] \in \mathbb{R}^{n \times N}$  be a matrix. The convex hull of the column vectors  $\{p_1, \dots, p_N\}$ , is denoted by

$$\begin{aligned} \mathcal{P}(p) &= \text{conv}(\{p_1, \dots, p_N\}) \\ &= \left\{ \sum_{i=1}^N \theta_i p_i \mid \exists \theta = (\theta_1, \dots, \theta_N)^\top \right. \\ &\quad \left. \theta \geq 0, 1^\top \theta = 1 \right\}. \end{aligned}$$

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The set  $\mathcal{P}(p) \subseteq \mathbb{R}^n$  is a polytope. Under the assumption that for all  $i \in \{1, \dots, N\}$  the vector  $p_i$  is not in the convex hull of  $\{p_1, \dots, p_N\} \setminus \{p_i\}$ , we call  $p$  the vertex matrix of the polytope  $\mathcal{P}(p)$ . Moreover, we use the notation

$$\mathcal{N}_i(p) = \{c \in \mathbb{R}^{n_x} \mid \forall k \neq i, (p_k - p_i)^\top c \leq 0\}$$

to represent the normal cone of  $\mathcal{P}(p)$  at its  $i$ -th vertex.

## II. PROBLEM SETTING

This paper concerns nonlinear control system of the form

$$\forall t \in \mathbb{R}, \quad \dot{x}(t) = f(x(t)) + Bu(t) + Cw(t). \quad (1)$$

Here,  $x \in W_{1,1}^{n_x}$  denotes the state trajectory, which is required to satisfy state constraints of the form

$$\forall t \in \mathbb{R}, \quad x(t) \in \mathbb{X} \subset \mathbb{R}^{n_x}.$$

The control input  $u \in L_1^{n_u}$  and the exogenous disturbance  $w \in L_1^{n_w}$  are assumed to be bounded by given sets,

$$\forall t \in \mathbb{R}, \quad u(t) \in \mathbb{U} \in \mathbb{K}_{\mathbb{C}}^{n_u} \quad \text{and} \quad w(t) \in \mathbb{W} \in \mathbb{K}_{\mathbb{C}}^{n_w}.$$

The function  $f: \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$  is assumed to be locally Lipschitz continuous, while  $B \in \mathbb{R}^{n_x \times n_u}$  and  $C \in \mathbb{R}^{n_x \times n_w}$  are given. The closed-loop reachable set of (1) for a given feedback law  $\mu: \mathbb{R} \times \mathbb{X} \rightarrow \mathbb{U}$  and initial condition  $x_0$  is denoted by

$$X(t, x_0, \mu) = \left\{ \xi_t \left| \begin{array}{l} \exists x \in W_{1,1}^{n_x}, \exists w \in L_1^{n_w} : \forall \tau \in [0, t] \\ \dot{x}(\tau) = f(x(\tau)) + B\mu(\tau, x(\tau)) + Cw(\tau) \\ x(0) = x_0, x(t) = \xi_t, w(\tau) \in \mathbb{W} \end{array} \right. \right\}.$$

Additionally, we recall the following standard definition of robust forward invariant tubes [15].

**Definition 1:** A set-valued function  $Y: [0, T] \rightarrow \mathbb{K}^{n_x}$  is an RFIT of (1) on  $[0, T]$ , if there exists a feedback control law  $\mu: [0, T] \times \mathbb{R}^{n_x} \rightarrow \mathbb{U}$  such that

$$Y(t_2) \supseteq \bigcup_{y_1 \in Y(t_1)} X(t_2 - t_1, y_1, \mu)$$

for all intervals  $[t_1, t_2] \subseteq [0, T]$ .

The goal of this paper is to solve Tube MPC problems of the form

$$\inf_{Y \in \mathcal{Y}} \int_0^T \mathcal{L}(Y(t)) dt \quad \text{s.t.} \quad \begin{cases} Y(t) \subseteq \mathbb{X} & \text{f.a. } t \in [0, T] \\ Y(0) = x_0 \end{cases} \quad (2)$$

with  $\mathcal{L}: \mathbb{K}^{n_x} \rightarrow \mathbb{R}$  denoting a stage cost. Here,  $\mathcal{Y}$  denotes the set of all RFITs of (1) on  $[0, T]$ . Notice that the current time of this MPC controller is set to 0.

Unfortunately, Problem (2) is, in general, computationally intractable. One of the main difficulties for solving (2) is that RFITs are set-valued functions which cannot be computed accurately in high dimensional state spaces. Therefore, our focus is on tractable and conservative approximations of (2) by optimizing over a family of set-valued functions with parameterized cross-sections. The following section focuses on the characterization of RFITs whose convex cross-sections are parameterized by their support function.

## III. MIN-MAX DIFFERENTIAL INEQUALITIES

This section focuses on providing a sufficient condition for characterizing the set of convex robust forward invariant tubes by means of min-max differential inequalities. Let us introduce the shorthand

$$\Gamma(c, v, \Psi) = \left\{ \begin{array}{l} f(\xi) + Bv + C\omega \\ c^\top \xi = V[\Psi](c) \\ \xi \in \Psi \\ \omega \in \mathbb{W} \end{array} \right\},$$

which is defined for all  $c \in \mathbb{R}^{n_x}$ , all  $v \in \mathbb{U}$ , and all  $\Psi \in \mathbb{K}^{n_x}$ . A proof of the following theorem can be found in [12].

**Theorem 1:** Let  $Y: [0, T] \rightarrow \mathbb{K}_{\mathbb{C}}^{n_x}$  be a set-valued function such that

- the function  $V[Y(\cdot)](c)$  is, for all  $c \in \mathbb{R}^{n_x}$ , Lipschitz continuous on  $[0, T]$ , and
- the set-valued function  $Y$  satisfies, for all  $c \in \mathbb{R}^{n_x}$  and all  $t \in [0, T]$ , the differential inequalities,

$$\forall t \in [0, T], \quad \dot{V}[Y(t)](c) \geq \min_{v \in \mathbb{U}} V[\Gamma(c, v, Y(t))](c) \quad (3)$$

with  $\dot{V}[Y(\cdot)](c) \in L_1^1$  denoting a weak derivative of  $V[Y(\cdot)](c)$  with respect to time.

Then,  $Y$  is an RFIT of (1) on  $[0, T]$ .

The sufficient conditions provided by Theorem 1 for an arbitrary convex set-valued function are computationally intractable in general. However, the min-max differential inequality can be checked constructively for certain parameterizations of the tube cross-sections, as shown for polytopic tubes next.

## IV. POLYTOPIC ROBUST FORWARD INVARIANT TUBES

The remaining sections are concerned with linear systems of the form

$$\forall t \in [0, T]: \dot{x}(t) = Ax(t) + Bu(t) + Cw(t), \quad (4)$$

where  $A \in \mathbb{R}^{n_x \times n_x}$  is a constant matrix.

**Remark 1:** The technical developments in this section can be extended to deal with nonlinear systems (1), by following the approach in [12].

In the following, we assume that the sets

$$\mathbb{U} = \mathcal{P}(\bar{u}) \quad \text{and} \quad \mathbb{W} = \mathcal{P}(\bar{w})$$

are polytopes with vertex matrices  $\bar{u} \in \mathbb{R}^{n_u \times m_u}$  and  $\bar{w} \in \mathbb{R}^{n_w \times m_w}$ , respectively. Our goal is to construct RFITs  $Y: [0, T] \rightarrow \mathbb{K}_{\mathbb{C}}^{n_x}$  with

$$Y(t) = \mathcal{P}(y(t)). \quad (5)$$

Here,  $y: [0, T] \rightarrow \mathbb{R}^{n_x \times N}$  a time-varying parameterization.

**Lemma 1:** The set-valued functions given by (5) satisfy the differential inequalities (3), if and only if the functions  $y: [0, T] \rightarrow \mathbb{R}^{n_x \times N}$  with  $y(t) = (y_1(t), \dots, y_N(t))$  satisfy

$$\forall c \in \mathcal{N}_i(y(t)), \quad c^\top \dot{y}_i(t) \geq \min_{v \in \mathbb{U}} \max_{\omega \in \mathbb{W}} c^\top (Ay_i(t) + Bv + C\omega)$$

for all  $t \in [0, T]$  and all  $i \in \{1, \dots, N\}$  as well as

$$y_i(t) \notin \text{conv} \left( \bigcup_{k \neq i} \{y_k(t)\} \right) \quad (6)$$

for all  $t \in [0, T]$  and all  $i \in \{1, \dots, N\}$ .

*Proof:* We start by introducing the shorthand notation

$$\Theta(c, p) = \underset{i}{\text{argmax}} c^\top p_i, \quad (7)$$

which is well-defined for any vertex matrix  $p \in \mathbb{R}^{n_x \times N}$  and any vector  $c \in \mathbb{R}^{n_x}$ . Notice that for any given  $c \in \mathbb{R}^{n_x}$ , (7) implies that there exists at least one  $i$  such that  $c \in \mathcal{N}_i(p)$ , since  $\Theta(c, p)$  is non-empty and we have  $c \in \mathcal{N}_i(p)$  for all  $i \in \Theta(c, p)$ . In other words, we must have

$$\bigcup_{i \in \{1, \dots, N\}} \mathcal{N}_i(p) = \mathbb{R}^{n_x}$$

for any given  $p \in \mathbb{R}^{n_x \times N}$ . Moreover, the implication

$$i \in \Theta(c, p) \implies V[\mathcal{P}(p)] = c^\top p_i$$

holds. Thus, the min-max differential inequality (3) for the set parameterization  $Y(t) = \mathcal{P}(y(t))$  can be written in the form

$$\forall c \in N_i(y(t)), \quad c^\top \dot{y}_i(t) = \min_{v \in \mathbb{U}} V[\Gamma(v, c, \mathcal{P}(y(t)))](c)$$

with

$$\begin{aligned} V[\Gamma(u, c, \mathcal{P}(y(t)))](c) &= \max_{x, w} c^\top (A\xi + Bv + C\omega) \\ \text{s.t.} \quad &\begin{cases} c^\top \xi = V[\mathcal{P}(y(t))](c) \\ \xi \in \mathcal{P}(y(t)) \\ w \in \mathbb{W}. \end{cases} \end{aligned} \quad (8)$$

Notice that all vertices  $y_i$  with  $i \in \Theta(c, y(t))$  are feasible points of the above maximization problem. Therefore, the differential inequality must hold at the vertices and we have

$$c^\top \dot{y}_i(t) \geq \min_{v \in \mathbb{U}} \max_{\omega \in \mathbb{W}} c^\top (Ay_i(t) + Bv + C\omega) \quad (9)$$

for all  $c \in N_i(y(t))$  and all  $i \in \{1, \dots, N\}$ .

The other way around, if (9) holds for all  $i \in \{1, \dots, N\}$ , we have

$$\begin{aligned} &\left\{ \xi \in \mathbb{R}^{n_x} \mid \begin{cases} c^\top \xi = V[\mathcal{P}(y(t))](c) \\ \xi \in \mathcal{P}(y(t)) \end{cases} \right\} \\ &= \bigcap_{i \in \Theta(c, y(t))} \mathcal{N}_i(y(t)) \\ &= \text{conv}(\{y_{k_1}(t), y_{k_2}(t), \dots, y_{k_m}(t)\}), \end{aligned} \quad (10)$$

as long as (6) holds. Here,  $\Theta(c, y(t)) = \{k_1, k_2, \dots, k_m\}$  is used to enumerate the indices in  $\Theta(c, y(t))$ . Substituting (10) in (8), yields

$$\begin{aligned} &V[\Gamma(v, c, \mathcal{P}(y(t)))](c) \\ &= \left( \max_{j \in \Theta(c, y(t))} c^\top Ay_{k_j}(t) \right) + c^\top Bu + \left( \max_{w \in \mathbb{W}} c^\top Cw \right). \end{aligned} \quad (11)$$

Let  $j^*(c)$  denote the maximizing index in (11). Then, (9) implies that

$$\dot{V}[\mathcal{P}(y(t))] \stackrel{(4),(11)}{=} c^\top \dot{y}_{k_{j^*(c)}}(t) \quad (12)$$

$$\stackrel{(9),(11)}{\geq} \min_{v \in \mathbb{U}} V[\Gamma(v, c, \mathcal{P}(y(t)))](c). \quad (13)$$

This shows the equivalence between (3) and (9).  $\blacksquare$

In order to establish computationally tractable conditions for verifying the conditions in Lemma 1, it is helpful to write the normal cones,  $\mathcal{N}_i(p)$ , in the form

$$\mathcal{N}_i(p) = \{c \mid G_i(p)c \leq 0\} \quad \text{with} \quad G_i(p) = \begin{pmatrix} (p_1 - p_i)^\top \\ \vdots \\ (p_{i-1} - p_i)^\top \\ (p_{i+1} - p_i)^\top \\ \vdots \\ (p_N - p_i)^\top \end{pmatrix}.$$

We now present the main result of this section, which is summarized in the next theorem.

**Theorem 2:** Let  $y = (y_1, \dots, y_N) : [0, T] \rightarrow \mathbb{R}^{n_x \times N}$  be any function satisfying

$$\begin{aligned} \forall t \in [0, T], j \in \{1, \dots, m_w\}, \\ \dot{y}_1(t) &= Ay_1(t) + Bu_1(t) + C\bar{w}_j - G_1(y(t))\lambda_{1,j}(t) \\ &\vdots \\ \dot{y}_N(t) &= Ay_N(t) + Bu_N(t) + C\bar{w}_j - G_N(y(t))\lambda_{N,j}(t) \end{aligned} \quad (14)$$

as well as

$$\forall t \in [0, T], \forall i \in \{1, \dots, N\}, \quad y_i(t) \notin \text{conv} \left( \bigcup_{k \neq i} \{y_k(t)\} \right)$$

for some functions  $u_i : [0, T] \rightarrow \mathbb{U}$  and  $\lambda_{i,j} : [0, T] \rightarrow \mathbb{R}_+^{(N-1)n_x}$  with  $(i, j) \in \{1, \dots, N\} \times \{1, \dots, m_w\}$ . Then, the set-valued function  $Y$  with  $Y(t) = \mathcal{P}(y(t))$  satisfies the differential inequality (3) for all  $[0, T]$ .

*Proof:* The first differential inequality condition from Lemma 1 holds if we have

$$\max_{\omega \in \mathbb{W}} \max_{c \in \mathcal{N}_i(y(t))} c^\top (Ay_i(t) + Bu_i(t) + C\omega - \dot{y}_i(t)) \leq 0$$

for at least one function  $u_i$ . Because the maximization problem on the left is bilinear in  $\omega$  and  $c$ , the corresponding maximizer for  $\omega$  must be a vertex of the polytope  $\mathcal{P}(\bar{w})$ . Thus, the above inequality is satisfied if

$$\begin{aligned} 0 &\geq \max_c c^\top (Ay_i(t) + Bu_i(t) + C\bar{w}_j - \dot{y}_i(t)) \\ \text{s.t.} \quad &G_i(y(t))c \leq 0 \end{aligned} \quad (15)$$

for all  $i \in \{1, \dots, N\}$  and all  $j \in \{1, \dots, m_w\}$ . By using duality in linear programming [24], it follows that the latter condition is equivalent to (14), where  $\lambda_{i,j}(t) \geq 0$  denotes the dual variables associated with the constraints in (15).  $\blacksquare$

## V. POLYTOPIC TUBE MPC

The results from the previous section can be used to derive tractable but conservative approximations of (2). An immediate consequence of Theorem 2 is that

$$\begin{aligned} & \inf_{y,u,\lambda} \int_0^T \mathcal{L}(\mathcal{P}(y(t))) dt \\ & \text{s.t.} \begin{cases} \forall t \in [0, T], \forall i \in \{1, \dots, N\}, \forall j \in \{1, \dots, n_w\}, \\ \dot{y}_i(t) = Ay_i(t) + Bu_j(t) + Cw_j - G_i(y(t))^\top \lambda_{i,j}(t) \\ \mathcal{P}(y(t)) \subseteq \mathbb{X}, u_i(t) \in \mathbb{U}, \lambda(t) \geq 0 \\ y_i(0) = x_0 \end{cases} \end{aligned} \quad (16)$$

yields a conservative approximation of (2). That is, if  $y$  is a solution of (16), then the set-valued function  $Y$  given by  $Y(t) = \mathcal{P}(y(t))$  is a feasible point of (2). Notice that if  $\mathbb{X}$  is given in the form

$$\mathbb{X} = \{\xi \in \mathbb{R}^{n_x} \mid h(x) \leq 0\}$$

for a convex (componentwise) function  $h$ , we have

$$\mathcal{P}(y(t)) \subseteq \mathbb{X} \quad \Leftrightarrow \quad \forall i \in \{1, \dots, N\}, \quad h(v_i) \leq 0$$

Similarly, if

$$\mathcal{L}(X) = \max_{x \in X} \ell(x)$$

is a worst case stage cost for a given convex function  $\ell$ , we have that

$$\mathcal{L}(\mathcal{P}(y(t))) = \max_i \ell(y_i(t)),$$

which serves as a computationally tractable reformulation of the objective function. In this case, (16) is a standard optimal control problem that can be solved with existing model predictive control tools. Notice, however, that choosing an objective function for Tube MPC schemes is a modeling problem on its own and that many different options are possible [13].

In practice, one implements the controller in closed-loop by sending the input

$$\mu_{\text{MPC}}(x_0) = u_i^*(0)$$

to the real process. Here,  $u_i^*(0)$  is the first element in the optimal control sequence with respect to the vertex  $i$ , given as part of the solution to Problem (16). In this context, it is irrelevant which vertex index  $i$  is picked—the feedback control law  $\mu_{\text{MPC}}$  is always robustly feasible, regardless of the specific choice of  $i$ . This follows from the fact that the vertices  $y_i(t)$  all coincide with  $x_0$  at  $t = 0$ .

**Remark 2:** *Recursive feasibility* can be guaranteed by introducing the additional constraint

$$Y(t+T) \subseteq Y_{\text{inv}}. \quad (17)$$

Here,  $Y_{\text{inv}} \subseteq \mathbb{X}$  is a robust forward invariant set for the control system (4).

## VI. NUMERICAL CASE STUDY

### A. Mathematical model

We consider system (4) with matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The control and disturbance sets are given by  $\mathbb{U} = [-8, 8]$  and  $\mathbb{W} = [-1, 1]$  respectively. The length of prediction horizon is set to  $T = 8$ , and the initial state of the system is  $x_0 = (0.3, 0.8)^\top$ .

The objective function for the Tube MPC problem is based on (16) and involves minimizing the functional

$$\int_0^T \mathcal{L}(\mathcal{P}(y(t))) dt$$

with

$$\mathcal{L}(\mathcal{P}(y(t))) = \sum_{i=0}^N \|y_i(t)\|_2^2 + \sum_{i=1}^N \sum_{j \neq i} \|y_i(t) - y_j(t)\|_2^2.$$

In addition, the state constraint

$$\mathcal{P}(y(t)) \subseteq \mathbb{X} = \{\xi \in \mathbb{R}^2 \mid \xi_1 \leq 0.65\}$$

is enforced.

### B. Numerical performance

Our numerical implementation is based on a single-shooting method with 20 control intervals using a Runge-Kutta integrator of order 4 with step-size  $h = 10^{-2}$ . Problem (16) was implemented in MATLAB, using YALMIP [25] with IPOPT as the underlying optimization solver [26]. The black dashed line in the left plot of Figure 1 shows a closed-loop simulation of a traditional model predictive controller for a particular disturbance scenario that leads to a constraint violation. Moreover, Figure 1 depicts selected cross-sections of the tube (with four vertices) in the  $x_1 - x_2$  space. The right plot of Figure 1 shows an optimized polytopic RFIT together with associated closed-loop simulations.

## VII. CONCLUSIONS

This paper has presented a practical approach to Tube MPC for uncertain control systems. The proposed approach relies on the construction of RFITs based on min-max differential inequalities. Here, one of the main contributions has been the derivation of a computationally tractable polytopic outer approximation of reachable tubes. The practical applicability of the associated polytopic Tube MPC controller has been illustrated by a numerical case study.

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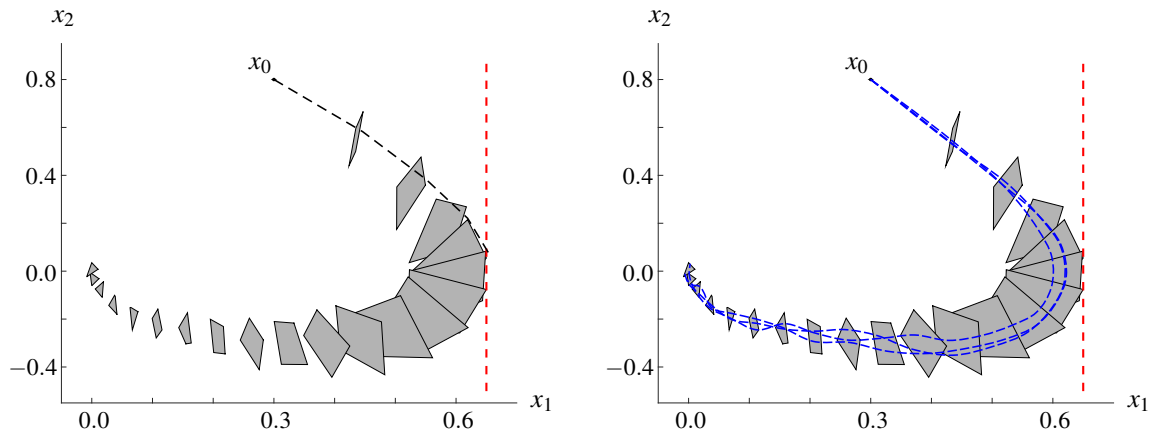


Fig. 1. The dark shaded regions show selected cross-sections of an optimized polytopic tube and the red dashed line shows the state constraint  $\mathbb{X} = \{y \mid y_1 \leq 0.65\}$ . Left: The black dashed line depicts the disturbed closed-loop trajectory for a traditional MPC controller, which violates the state constraint. Right: The blue dashed lines show closed-loop trajectories for three selected uncertainty realizations.

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