Min-max Differential Inequalities for Polytopic Tube MPC

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Abstract— **This paper is about Tube MPC for input affine nonlinear control systems, influenced by bounded and time-varying additive disturbances. In particular, we consider tube MPC formulations based on robust forward invariant tubes (RFITs) and their convex approximations. We present sufficient and necessary conditions, in the form of a pair of min-max/max-min differential inequalities, for an RFIT to be tight. The paper also presents a polytopic outer parameterization and inner approximation scheme for RFITs based on a vertex representation of the polytopes. The results are illustrated with a numerical case study.**

I. Introduction

Robust model predictive control (MPC) solves at every sample instant an optimal control problem, whose decision variables are future control policies, i.e. the feedback control functions, which guarantee constraint satisfaction in the presence of model uncertainties or external disturbances.

Robust MPC based on parametric set-propagation is also known as tube MPC [1]. Tube MPC for linear discrete-time system has been applied in a variety of settings, e.g. these techniques have in common fact that they parameterize the feedback control law offline [2], [3]. In terms of online design, a synthesis approach for tackling linear discrete-time system is developed in [4]. The proposed technique allows the simultaneous online optimization of tubes as well as the control policy for those problems with tracking objectives. Its extension for dealing with more general cost functions and for reducing the computational complexity are presented in [5] and [6], respectively.

Other state-of-the-art approaches in designing feedback control scheme adopts a vertex control perspective. Vertex control technique constructs a linear state feedback controller, given in terms of the controls at the vertices of the polyhedral state constraint set. These methods find their origins in Gutman's work [7]. Recent advances based on an interpolating scheme between global vertex control and local unconstrained robust optimal control can be found in [8], [9]. However, the above articles only deal with deterministic system. For time-varying uncertain plant, we recommend to refer the paper [10], which investigated the worst case minimal reaching time.

A recent proposed technique formulated the tube MPC problem via min-max differential inequalities (DIs), which exploits the properties on the boundary of robust forward invariant tubes (RFITs) instead of parameterizing the feedback tself [11]. Here, RFIT is a set-valued function, which encloses all possible state trajectories under a given feedback control law. A practical implementation of such controller is based on an ellipsoidal parameterization of tubes cross-sections. However, the ellipsoidal parameterization unavoidably leads to the conservatism of the predicted tube. Even for linear systems, the tractable RFIT is usually not an ellipsoid.

Contribution: In order to fix aforementioned issues, we proposed to parameterize RFITs with polytopes via min-max DIs. The advantage for exploiting the polytopic parameterization is that we can reach an arbitrary precision insofar as the tubes are represented with sufficient accuracy. Specifically, this can be achieved by increasing the number of polytope's vertices.

Outline: The remainder of the paper is organized as follows, Section II introduces the problem formulation. Section III introduces the characterization of an RFIT via differential inequalities. The parameterization of RFITs with polytopic cross-sections is presented in Section IV. Section V presents a practical implementation of tube-based MPC scheme for linear system based on the results, which are developed in the previous sections. A numerical case study is given in Section VI. Section VII concludes the paper.

Notation: The set of n-dimensional Lebesgue integrable functions is is denoted by L_1^n. Its associated Sobolev space of weakly differentiable functions with L_1^n derivatives is denoted by $W_{1,1}^n$. We denote by $\mathbb{K}^n$ and $\mathbb{K}_C^n$ the sets of compact and convex compact sets in $\mathbb{R}^n$, respectively. Let $Z \in \mathbb{K}_C^n$, its support function, $V[Z] : \mathbb{R}^n \to \mathbb{R}$, is given by

$$\forall c \in \mathbb{R}^n, \quad V[Z](c) := \max_{z \in Z} \; c^\mathsf{T} z \, .$$

The Hausdorff distance between $Y, Z \subseteq R^n$ is denoted by

$$d_\mathrm{H}(Y,Z) = \max \left\{ \sup_{y\in Y} \inf_{z \in Z} \|y-z\|, \sup_{z\in Z} \inf_{y \in Y} \|y-z\| \right\} .$$

Let $p = [p_1, \ldots, p_N] \in \mathbb{R}^{n \times N}$ be a matrix. The convex hull of $\{p_1, \ldots, p_N\}$, is denoted by

$$\begin{aligned} \mathscr{P}(p) &= \mathrm{conv}\left(\{p_1, \ldots, p_k\}\right) \\ &= \left\{ \sum_{i=1}^N \theta_i p_i \;\middle|\; \begin{array}{c} \exists \theta = (\theta_1, \ldots, \theta_N)^\mathsf{T} \\ \theta \geq 0 \,,\; 1^\mathsf{T}\theta = 1 \end{array} \right\} . \end{aligned}$$

The set $\mathscr{P}(p) \subseteq \mathbb{R}^n$ can be interpreted as a polytope. Under the assumption that no p_i is in the convex hull of $\{p_1, \ldots, p_N\} \setminus \{p_i\}$, we call p the vertex matrix of $\mathscr{P}(p)$. Moreover, we use the notation

$$\mathscr{N}_i(p) = \{ c \in \mathbb{R}^{n_x} \mid \forall k \neq i, \; (p_k - p_i)^\mathsf{T} c \leq 0 \, \}$$

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to denote the normal cone of $\mathscr{P}(p)$ at its i-th vertex.

II. Problem Setting

This paper concerns nonlinear control system of the form

$$\forall t \in \mathbb{R}, \quad \dot{x}(t) = f(x(t)) + Bu(t) + Cw(t) \; . \tag{1}$$

Here, $x \in W_{1,1}^{n_x}$ denotes the state trajectory, which is required to satisfy state constraints of the form

$$\forall t \in \mathbb{R}, \quad x(t) \in \mathbb{X} \subset \mathbb{R}^{n_x} \; .$$

The control input $u \in L_1^{n_u}$ and the exogenous disturbance $w \in L_1^{n_w}$ are assumed to be bounded by some given sets, i.e.

$$\forall t \in \mathbb{R}, \quad u(t) \in \mathbb{U} \in \mathbb{K}_{\mathrm{C}}^{n_u} \quad \text{and} \quad w(t) \in \mathbb{W} \in \mathbb{K}_{\mathrm{C}}^{n_w} \; .$$

The function $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is assumed to be locally Lipschitz continuous, while $B \in \mathbb{R}^{n_x \times n_u}$ and $C \in \mathbb{R}^{n_x \times n_w}$ are given. The closed-loop reachable set of (1) for a given feedback law $\mu : \mathbb{R} \times \mathbb{X} \to \mathbb{U}$ and initial condition x_0 is denoted by

$$X(t, x_0, \mu) = \left\{ \xi_t \left| \begin{array}{l} \exists x \in W_{1,1}^{n_x}, \; \exists w \in L_1^{n_w} : \; \forall \tau \in [0,t] \\ \dot{x}(\tau) = f(x(\tau)) + B\mu(\tau, x(\tau)) + Cw(\tau) \\ x(0) = x_0 \, , \; x(t) = \xi_t \, , \; w(\tau) \in \mathbb{W} \end{array} \right. \right\} .$$

Additionally, we recall the following standard definition of robust forward invariant tubes [1].

Definition 1: A set-valued function $Y : [0,T] \to \mathbb{K}^{n_x}$ is an RFIT of (1) on $[0,T]$, if there exists an feedback control law $\mu : [0,T] \times \mathbb{R}^{n_x} \to \mathbb{U}$ such that

$$Y(t_2) \supseteq \bigcup_{y_1 \in Y(t_1)} X(t_2 - t_1, y_1, \mu)$$

for all intervals $[t_1, t_2] \subseteq [0,T]$.

The goal of this paper is to solve tube MPC problems of the form

$$\inf_{Y \in \mathscr{Y}} \int_0^T \mathscr{L}(Y(t)) \mathrm{d}t \quad \text{s.t.} \begin{cases} Y(t) \subseteq \mathbb{X} & \forall t \in [0,T] \\ Y(0) = x_0 \end{cases} \tag{2}$$

with $\mathscr{L} : \mathbb{K}^{n_x} \to \mathbb{R}$ denoting a stage cost. Here, $\mathscr{Y}$ denotes the set of all RFITs of (1) on $[0,T]$. Notice that the current time of this MPC controller is continuously reset to 0, i.e., such that the current prediction horizon is equal to $[0,T]$; see also [12] for more details regarding tube MPC.

Unfortunately, Problem (2) is, in general, computationally intractable. One of the main difficulties for solving (2) is that RFITs are set-valued functions which cannot be computed accurately in high dimensional state spaces. Therefore, our focus is on tractable outer approximations of RFITs. The following section analyzes a convex outer approximation scheme for RFITs.

III. Differential inequalities

This section focuses on providing a characterization of the set of convex robust forward invariant tubes by means of differential inequalities. First, let us introduce the shorthand

$$\Gamma(c, \nu, \Psi) = \left\{ f(\xi) + B\nu + C\omega \left| \begin{array}{c} c^\mathsf{T} \xi = V[\Psi](c) \\ \xi \in \Psi \\ \omega \in \mathbb{W} \end{array} \right. \right\} ,$$

which is defined for all $c \in \mathbb{R}^{n_x}$, all $\nu \in \mathbb{U}$, and all $\Psi \in \mathbb{K}^{n_x}$. The next theorem provides sufficient conditions for a set-valued function to be an RFIT of (1) on an interval $[0,T]$.

Theorem 1: Let $Y : [0,T] \to \mathbb{K}_{\mathrm{C}}^{n_x}$ be a set-valued function such that

- the function $V[Y(\cdot)](c)$ is, for all $c \in \mathbb{R}^{n_x}$, Lipschitz continuous on $[0,T]$, and
- the set-valued function Y satisfies, for all $c \in \mathbb{R}^{n_x}$ and all $t \in [0,T]$, the differential inequalities,

$$\forall t \in [0,T], \quad \dot{V}[Y(t)](c) \geq \min_{\nu \in \mathbb{U}} V[\Gamma(c, \nu, Y(t))](c) \; . \tag{3}$$

Then, Y is an RFIT of (1) on $[0,T]$.

Proof: See [11] for a proof. ■

Since RFITs are not unique, one is often interested in knowing whether an RFIT is *tight*.

Definition 2: An RFIT $Y : [0,T] \to \mathbb{K}^{n_x}$ is called tight on the time interval $[0,T]$, if for any RFIT $Z : [0,T] \to \mathbb{K}^{n_x}$ satisfying $Z(0) = Y(0)$ and $Z(t) \subseteq Y(t)$ for all $t \in (0,T]$, it follows that $Z(t) = Y(t)$ for all $t \in [0,T]$.

In the remainder of this section we analyze necessity of the conditions of RFIT conditions provided by Theorem 1 in order to establish a tightness result for RFITs. For this aim, we introduce the shorthand

$$\Lambda(c, \omega, \Psi) = \left\{ f(\xi) + B\nu + C\omega \left| \begin{array}{c} c^\mathsf{T} \xi = V[\Psi](c) \\ \xi \in \Psi \\ \nu \in \mathbb{U} \end{array} \right. \right\} ,$$

which is defined for all $c \in \mathbb{R}^{n_x}$, all $\omega \in \mathbb{W}$, and all $\Psi \in \mathbb{K}^{n_x}$.

Theorem 2: Let $Y : [0,T] \to \mathbb{K}_{\mathrm{C}}^{n_x}$ be a tight RFIT for (1) on $[0,T]$. Let $Z : [0,T] \to \mathbb{K}_{\mathrm{C}}^{n_x}$ be a set-valued function such that

- the function $V[Z(\cdot)](c)$ is, for all $c \in \mathbb{R}^{n_x}$, Lipschitz continuous on $[0,T]$, and
- the set-valued function Z satisfies, for all $c \in \mathbb{R}^{n_x}$ and all $t \in [0,T]$, the differential inequalities,

$$\dot{V}[Z(t)](c) \leq \max_{\omega \in \mathbb{W}} -V[-\Lambda(c, \omega, Z(t))](c) \tag{4}$$

$$V[Z(0)](c) \leq V[Z(0)](c) \; . \tag{5}$$

Then, $Z(t) \subseteq Y(t)$ for all $[0,T]$.

Proof: Let $t \in [0,T]$ be any given (fixed) time. Now, consider the reverse dynamic system

$$\forall \tau \in [0, T-t], \quad \dot{\xi}(\tau) = -f(\xi(\tau)) - Bu(\tau) - Cw(\tau), \tag{6}$$

with $\xi(0) = \xi_0$. Here, Z satisfies $Z(t) \subseteq Y(t)$ if Z is an RFIT of (6) on $[0, T-t]$ with the roles of u and w interchanged. In other words, as our goal is to establish the inclusion $Z(t) \subseteq Y(t)$, it is sufficient to ensure that there exists a feedback control law $\overline{\mu} : [0, T-t] \times \mathbb{R}^{n_x} \to \mathbb{W}$, such that any closed-loop solution of (6) with $w(\tau) = \overline{\mu}(\tau, \xi(\tau))$ satisfies $z(t) \in Y(0)$ for any $u : [0, T-t] \to \mathbb{U}$.

Now, Theorem 1 implies that if the set-valued function $\Psi : [0, T-t] \to \mathbb{K}_{\mathrm{C}}^{n_x}$ is such that $\dot{V}[\Psi(\cdot)](c)$ is, for all $c \in \mathbb{R}^{n_x}$, Lipschitz continuous on $[0, T-t]$ and satisfies

$$\dot{V}[\Psi(\tau)](c) \geq \min_{\omega \in \mathbb{W}} V[-\Lambda(c, \omega, \Psi(\tau))](c) \tag{7}$$

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for all $\tau \in [0, T-t]$ and all $c \in \mathbb{R}^{n_x}$, then Ψ is an RFIT for (6) on the time interval $[0, T-t]$—with the roles of u and w interchanged. Additionally, we enforce

$$V[\Psi(0)](c) \leq V[Y(0)](c) ,$$

such that $\Psi(0) \subseteq Y(0)$, i.e., such that the desired inclusion $z(t) \in Y(0)$ can be guaranteed. The statement of the theorem follows after setting $Z(t) = \Psi(T-t)$ in (7) and multiplying the resulting differential inequality with -1. ■

The next corollary summarizes one of the main theoretical contributions of this paper, namely necessary and sufficient conditions for an RFIT to be tight.

Corollary 1: Let $Y : [0,T] \to \mathbb{K}_{\mathrm{C}}^{n_x}$ be an RFIT of (1) on $[0,T]$ such that its images have nonempty interior and $V[Y(\cdot)](c)$ is, for all $c \in \mathbb{R}^{n_x}$, Lipschitz continuous on $[0,T]$. Then, Y is tight if and only if it satisfies the differential inequalities (3) and (4) for all $c \in \mathbb{R}^{n_x}$ and all $t \in [0,T]$.

Proof: If Y satisfies (3) and (4) the statements of Theorems 1 and 2 imply that Y must be a tight RFIT. Thus, conditions (3) and (4) are sufficient to establish that Y is tight. However, establishing necessity, is somewhat less trivial.

We first introduce the auxiliary assumption that $Y(t)$ is strictly convex. In this case, the supporting facet of $Y(t)$ in the direction c is a singleton, i.e.

$$\left\{ \xi \in \mathbb{R}^{n_x} \middle| \begin{array}{c} c^\intercal \xi = V[Y(t)](c) \\ \xi \in Y(t) \end{array} \right\} = \underset{\xi \in Y(t)}{\operatorname{argmax}} \ c^\intercal \xi = \{ \xi^\star(c, Y(t)) \} \ .$$

Moreover, since $Y(t)$ has a nonempty interior, it follows that $\xi^\star(\cdot, Y(t))$ is injective. Thus, the differential inequalities (3) and (4) can be written as

$$\begin{aligned} \dot{V}[Y(t)](c) &\geq \min_{\nu \in \mathbb{U}} \max_{\omega \in \mathbb{W}} c^\intercal \left(f\left(\xi^\star(c, Y(t))\right) + B\nu + C\omega \right) \\ &= c^\intercal f\left(\xi^\star(c, Y(t))\right) + r(c) , \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{V}[Y(t)](c) &\leq \max_{\omega \in \mathbb{W}} \min_{\nu \in \mathbb{U}} \left(f\left(\xi^\star(c, Y(t))\right) + B\nu + C\omega \right) \\ &= c^\intercal f\left(\xi^\star(c, Y(t))\right) + r(c) , \end{aligned} \tag{9}$$

where the scalar offset

$$r(c) = \left(\min_{\nu \in \mathbb{U}} c^\intercal B \nu \right) + \left(\max_{\omega \in \mathbb{W}} c^\intercal C \omega \right)$$

is time-invariant and depends on c only. This splitting is possible, because the control u and uncertainty w enter affinely. It remains to show that any RFIT Y with strictly convex images, such that $V[Y(\cdot)](c)$ is Lipschitz continuous, satifies the differential equation

$$\dot{V}[Y(t)](c) = c^\intercal f\left(\xi^\star(c, Y(t))\right) + r(c) \tag{10}$$

for all $c \in \mathbb{R}^{n_x}$ and all $t \in [0,T]$. Using the same arguments as in the proof of [11, Theorem 2], one can obtain the following inequality

$$\begin{aligned} \dot{V}[Y(\tau)](c) &= c^\intercal \frac{\partial}{\partial t} \xi^\star(c, Y(t)) \\ &= \lim_{h \to 0} \frac{c^\intercal \xi^\star(c, Y(t+h)) - c^\intercal \xi^\star(c, Y(t))}{h} \\ &\leq \min_{\nu \in \mathbb{U}} \max_{\omega \in \mathbb{W}} c^\intercal \left(f\left(\xi^\star(c, Y(t))\right) + Bu + Cw \right) \\ &= c^\intercal f\left(\xi^\star(c, Y(t))\right) + r(c) \ . \end{aligned}$$

At this point it is important to re-iterate that the map $\xi^\star(\cdot, Y(t))$ is injective, such that we can ensure that the constructed boundary control law is well-defined. Next, we reverse the dynamic system and repeat the same arguments used until now to show that the associated reverse inequality,

$$\dot{V}[Y(\tau)](c) \geq c^\intercal f\left(\xi^\star(c, Y(t))\right) + r(c) ,$$

also holds whenever Y is a tight RFIT. In other words the statement of the corollary holds under the additional assumption that $Y(t)$ is strictly convex.

It remains to show that the statement of the corollary still holds after removing the assumption that $Y(t)$ is strictly convex. Here, the main idea is to first approximate the convex tube Y with a strictly convex tube such that the above statement applies. Then, the limit for vanishing approximation errors is taken. The technical details explaining how to take this limit can be found in [13]. The corresponding method can be applied one-to-one to conclude the statement of this corrollary. ■

IV. Polytopic Inner and Outer Approximations

The remaining sections are concerned with linear systems of the form

$$\forall t \in [0,T] : \ \dot{x}(t) = Ax(t) + Bu(t) + Cw(t), \tag{11}$$

where $A \in \mathbb{R}^{n_x \times n_x}$ is a constant matrix. We shift focus to linear time invariant systems, as their closed-loop reachable sets inherently convex. Thus, any tight RFIT will also be convex [1].

At this point, we should mention that similar methods to those presented in this section can also be constructed for nonlinear systems (1). In this case, the system can be linearized and the nonlinearity be considered as an additive disturbance bounded within some nonlinearity estimate [11]. Although, for such a system, our tightness results would be, unfortunately, lost.

In the following, we assume that the sets

$$\mathbb{U} = \mathscr{P}(\overline{u}) \quad \text{and} \quad \mathbb{W} = \mathscr{P}(\overline{w})$$

are polytopes with vertex matrices $\overline{u} \in \mathbb{R}^{n_u \times m_u}$ and $\overline{w} \in \mathbb{R}^{n_w \times m_w}$ respectively. Our goal is to construct set-valued functions $Y, Z : [0,T] \to \mathbb{K}_{\mathrm{C}}^{n_x}$ with

$$Y(t) = \mathscr{P}(y(t)) \quad \text{and} \quad Z(t) = \mathscr{P}(z(t)) \tag{12}$$

which are, respectively, inner and outer approximations of a tight RFIT $\widetilde{X}$ on $[0,T]$. In other words, such that they satisfy $Z(t) \subseteq \widetilde{X}(t) \subseteq Y(t)$ for all $t \in [0,T]$. Here, $y, z : [0,T] \to \mathbb{R}^{n_x \times N}$ denote the time-varying vertex matrices of the inner and outer polytopic approximations. In the remainder of this section, we impose the following blanket assumption.

***Assumption** 1:* The vertex matrices $y(t)$ and $z(t)$ are, for all $t \in [0,T]$, minimal parameterizations of $\mathscr{P}(y(t))$ and $\mathscr{P}(z(t))$, respectively. In other words, we have

$$\begin{array}{ll} \forall t \in [0,T], & y_i(t) \notin \{y_{1(t),\ldots,y_N(t)}\} \setminus \{y_i(t)\} \\ \text{and} & z_i(t) \notin \{z_{1(t),\ldots,z_N(t)}\} \setminus \{y_i(t)\} \end{array}$$

for all $i \in \{1,\ldots,N\}$.

***Lemma** 1:* The set-valued functions given by

$$Y(t) = \mathscr{P}(y(t)) \quad \text{and} \quad Z(t) = \mathscr{P}(z(t))$$

satisfy the differential inequalities (3) and (4), respectively, if and only if the functions $y,z : [0,T] \to \mathbb{R}^{n_x \times N}$ with $y(t) = (y_1(t),\ldots,y_N(t))$ and $z(t) = z_1,\ldots,z_N$ satisfy

$$\begin{aligned} \forall c \in \mathscr{N}_i(y(t)),\ c^\mathsf{T}\dot{y}_i(t) &\geq \min_{\nu\in\mathbb{U}} \max_{\omega\in\mathbb{W}} c^\mathsf{T}\left(Ay_i(t)+B\nu+C\omega\right) \\ \forall c \in \mathscr{N}_i(z(t)),\ c^\mathsf{T}\dot{z}_i(t) &\leq \max_{\omega\in\mathbb{W}} \min_{\nu\in\mathbb{U}} c^\mathsf{T}\left(Az_i(t)+B\nu+C\omega\right) \end{aligned}$$

for all $t \in [0,T]$ and all $i \in \{1,\ldots,N\}$.

Proof: We start by introducing the shorthand notation

$$\Theta(c,p) = \operatorname*{argmax}_i c^\mathsf{T} p_i \,, \tag{13}$$

which is well-defined for any vertex matrix $p \in \mathbb{R}^{n_x \times N}$ and any vector $c \in \mathbb{R}^{n_x}$. Notice that for any given $c \in \mathbb{R}^{n_x}$, (13) implies that there exists at least one i such that $c \in \mathscr{N}_i(p)$, since $\Theta(c,p)$ is non-empty and we have $c \in \mathscr{N}_i(p)$ for all $i \in \Theta(c,p)$. In other words, we must have

$$\bigcup_{i\in\{1,\ldots,N\}} \mathscr{N}_i(p) = \mathbb{R}^{n_x}$$

for any given $p \in \mathbb{R}^{n_x \times N}$. Moreover, the implication

$$i \in \Theta(c,p) \quad \Longrightarrow \quad V[\mathscr{P}(p)] = c^\mathsf{T} p_i$$

holds. Thus, the min-max differential inequality (3) for $Y(t) = \mathscr{P}(y(t))$ can be written in the form

$$\forall c \in N_i(y(t)), \quad c^\mathsf{T}\dot{y}_i(t) = \min_{\nu\in\mathbb{U}} V\left[\Gamma(\nu,c,\mathscr{P}(y(t)))\right](c)$$

with

$$\begin{aligned} V[\Gamma(u,c,\mathscr{P}(y(t)))](c) = \max_{x,w}\ & c^\mathsf{T}(A\xi + B\nu + C\omega) \\ \text{s.t.} & \left\{ \begin{array}{l} c^\mathsf{T}\xi = V[\mathscr{P}(y(t))](c) \\ \xi \in \mathscr{P}(y(t)) \\ w \in \mathbb{W}. \end{array} \right. \end{aligned} \tag{14}$$

Notice that all vertices y_i with $i \in \Theta(c,y(t))$ are feasible points of the above maximization problem. Thus, the min-max differential inequality must hold at the vertices and we have

$$c^\mathsf{T}\dot{y}_i(t) \quad \geq \quad \min_{\nu\in\mathbb{U}} \max_{\omega\in\mathbb{W}} c^\mathsf{T}\left(Ay_i(t)+B\nu+C\omega\right) \tag{15}$$

for all $c \in N_i(y(t))$ and all $i \in \{1,\ldots,N\}$.

The other way around, if (15) holds for all $i \in \{1,\ldots,N\}$, we have

$$\begin{aligned} &\left\{ \xi \in \mathbb{R}^{n_x} \,\middle|\, \begin{array}{l} c^\mathsf{T}\xi = V[\mathscr{P}(y(t))](c) \\ \xi \in \mathscr{P}(y(t)) \end{array} \right\} \\ &\qquad = \bigcap_{i\in\Theta(c,y(t))} \mathscr{N}_i(y(t)) \\ &\qquad = \operatorname{conv}\left(\{y_{k_1}(t), y_{k_2}(t),\ldots,y_{k_m}(t)\}\right) . \end{aligned} \tag{16}$$

Here, $\Theta(c,y(t)) = \{k_1,k_2,\ldots,k_m\}$ is used to enumerate the indices in $\Theta(c,y(t))$. Substituting (16) in (14), yields

$$\begin{aligned} &V[\Gamma(\nu,c,\mathscr{P}(y(t)))](c) \\ &\quad = \left(\max_{j\in\Theta(c,y(t))} c^\mathsf{T}Ay_{k_j}(t) \right) + c^\mathsf{T}Bu + \left(\max_{w\in\mathbb{W}} c^\mathsf{T}Cw \right) . \end{aligned} \tag{17}$$

Let $j^\star(c)$ denote the maximizing index in (17). Then, (15) implies that

$$\begin{aligned} \dot{V}[\mathscr{P}(y(t))] &\overset{(11),(17)}{=} c^\mathsf{T}\dot{y}_{k_{j^\star(c)}}(t) & (18) \\ &\overset{(15),(17)}{\geq} \min_{\nu\in\mathbb{U}} V[\Gamma(\nu,c,\mathscr{P}(y(t)))](c) \,. & (19) \end{aligned}$$

This shows the equivalence between (3) and (15). The equivalence between the second differential inequality in this lemma and (4) can be proven using analogous arguments after replacing all maximization problems with minimization problems and vice versa. ■

In order to establish a computationally tractable condition for verifying the conditions of Lemma 1, it is helpful to write the normal cones, $\mathscr{N}_i(p)$, in the form

$$\mathscr{N}_i(p) = \{c \mid G_i(p)c \leq 0\} \text{ with } G_i(p) = \begin{pmatrix} (p_1 - p_i)^\mathsf{T} \\ \vdots \\ (p_{i-1} - p_i)^\mathsf{T} \\ (p_{i+1} - p_i)^\mathsf{T} \\ \vdots \\ (p_N - p_i)^\mathsf{T} \end{pmatrix} .$$

The main result of this section is summarized next.

***Theorem** 3:* Let $y = (y_1,\ldots,y_N) : [0,T] \to \mathbb{R}^{n_x \times N}$ be any functions satisfying

$$\begin{aligned} &\forall t \in [0,T],\ j \in \{1,\ldots,m_w\}, \\ &\quad \dot{y}_1(t) = Ay_1(t) + Bu_1(t) + C\overline{w}_j - G_1(y(t))\lambda_{1,j}(t) \\ &\quad\quad \vdots \\ &\quad \dot{y}_N(t) = Ay_N(t) + Bu_N(t) + C\overline{w}_j - G_N(y(t))\lambda_{N,j}(t) \end{aligned} \tag{20}$$

for some functions $u_i : [0,T] \to \mathbb{U}$ and $\lambda_{i,j} : [0,T] \to \mathbb{R}_+^{(N-1)n_x}$ with $(i,j) \in \{1,\ldots,N\} \times \{1,\ldots m_w\}$. Then, the set-valued function Y with $Y(t) = \mathscr{P}(y(t))$ satisfies the differential inequality (3) for all $[0,T]$.

Similarly, Let $z = (z_1, \ldots, z_N) : [0,T] \to \mathbb{R}^{n_x \times N}$ be any functions satisfying

$$\begin{aligned} &\forall t \in [0,T],\, j \in \{1,\ldots,m_u\}, \\ &\quad \dot{z}_1(t) = Az_1(t) + B\overline{u}_j + Cw_1(t) - G_1(y(t))\kappa_{1,j}(t) \\ &\quad \vdots \\ &\quad \dot{z}_N(t) = Az_N(t) + B\overline{u}_j + Cw_N(t) - G_1(y(t))\kappa_{N,j}(t) \end{aligned} \tag{21}$$

for some functions $w_i : [0,T] \to \mathbb{W}$ and $\kappa_{i,j} : [0,T] \to \mathbb{R}_+^{(N-1)n_x}$ with $(i,j) \in \{1,\ldots,N\} \times \{1,\ldots m_u\}$. Then, the set-valued function Z with $Z(t) = \mathscr{P}(z(t))$ satisfies the differential inequality (4) for all $[0,T]$.

Proof: The first differential inequality condition from Lemma 1 holds if we have

$$\max_{\omega \in \mathbb{W}} \max_{c \in \mathscr{N}_i(y(t))} c^\intercal (Ay_i(t) + Bu_i(t) + C\omega - \dot{y}_i(t)) \leq 0$$

for at least one function u_i. Because the maximization problem on the left is bilinear in ω and c, the corresponding maximizer for ω must be a vertex of the polytope $\mathscr{P}(\overline{w})$. Thus, the above inequality is satisfied if

$$\begin{aligned} 0 \geq \max_c \;& c^\intercal (Ay_i(t) + Bu_i(t) + C\overline{w}_j - \dot{y}_i(t)) \\ \text{s.t.} \;& G_i(y(t))c \leq 0 \end{aligned} \tag{22}$$

for all $i \in \{1,\ldots,N\}$ and all $j \in \{1,\ldots,m_w\}$. By using duality in linear programming [14], it follows that the latter condition is equivalent to (20), where $\lambda_{i,j}(t) \geq 0$ denotes the dual variables associated with the constraints in (22).

The derivation of the condition (21) is then analogous and can be obtained directly by exchanging the roles of u and w recalling that $\overline{u}$ denotes the vertices of the polytope $\mathbb{U}$. ■

V. Polytopic Tube MPC

The considerations from the previous section can be used to derive tractable but conservative approximations of (2). An immediate consequence of Theorem 3 is that

$$\begin{aligned} \inf_{y,u,\lambda} \;& \int_0^T \mathscr{L}(\mathscr{P}(y(t)))\,\mathrm{d}t \\ \text{s.t.} \;& \begin{cases} \forall t \in [0,T],\, \forall i \in \{1,\ldots,N\}, \forall j \in \{1,\ldots,n_w\}, \\ \dot{y}_i(t) = Ay_i(t) + Bu_i(t) + Cw_j - G_i(y(t))^\intercal \lambda_{i,j}(t) \\ \mathscr{P}(y(t)) \subseteq \mathbb{X},\, u_i(t) \in \mathbb{U},\, \lambda(t) \geq 0 \\ y_i(0) = x_0 \end{cases} \end{aligned} \tag{23}$$

yields a conservative approximation of (2). That is, if y is a solution of (23), then the set-valued function Y given by $Y(t) = \mathscr{P}(y(t))$ is a feasible point of (2). Notice that if $\mathbb{X}$ is given in the form

$$\mathbb{X} = \{\xi \in \mathbb{R}^{n_x} \mid h(x) \leq 0\}$$

for a convex (componentwise) function h, we have

$$\mathscr{P}(y(t)) \subseteq \mathbb{X} \qquad \Leftrightarrow \qquad \forall i \in \{1,\ldots,N\}, \quad h(v_i) \leq 0$$

Similarly, if

$$\mathscr{L}(X) = \max_{x \in X} \ell(x)$$

is a worst case stage cost for a given convex function ℓ, we have

$$\mathscr{L}(\mathscr{P}(y(t))) = \max_i \ell(y_i(t)) ,$$

i.e., a computationally tractable reformulation of the objective function is avaliable, too. In this case, (23) is a standard optimal control problem that can be solved with existing model preditictive control software. In practice, one implements the closed-loop by sending the input

$$\mu_{\mathrm{MPC}}(x_0) = u_i^\star(0)$$

to the real process. Here, $u_i^\star(0)$ denotes an optimal solution of (23) for the control u_i. In this context, it is irrelevant which vertex index i is picked—the feedback law μ_{MPC} is always robustly feasible, indepently of the choice of i. This follows from the fact that the vertices $y_i(t)$ all coincide with x_0 at $t = 0$.

Another issue should be considered here is sometimes controls may overpower the influence of disturbances, which may lead to the degeneration during the evolution of the polytopic tube. In order to fix this problem, we introduce the additional constraint $(*)$ with a tunning parameter $\varepsilon > 0$.

Recursive feasibility can be enforced by imposing the additional constraint

$$Y(t+T) \subseteq Y_{\mathrm{inv}} , \tag{24}$$

with $Y_{\mathrm{inv}} \subseteq \mathbb{X}$ being a robust forward invariant set for the control system (11). Multiple ways for constructing such sets can be found in the literature. For example, in [11] a method for computing ellipsoidal robust forward invariant sets for ensuring recursive feasibility of a tube MPC scheme was proposed.

VI. Numerical Case Study

We consider system (11) with matrices

$$A = \begin{pmatrix} 0 & 1 \\ -0.2 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \text{and} \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

The control and disturbance sets are given by $\mathbb{U} = [-8,8]$ and $\mathbb{W} = [-1,1]$ respectively. The length of prediction horizon is set to $T = 8$, and the initial state of the system is $x_0 = (0.3, 0.85)^\intercal$.

The objective function for the Tube MPC problem (23) was chosen as

$$\int_0^T \left(\mathscr{L}(\mathscr{P}(y(t)))) + \sum_{i=1}^N \|u_i(t)\|_2^2 + \sum_{i=1}^N \sum_{j=1}^{n_w} \|\lambda_{i,j}(t)\|_2^2 \right) \mathrm{d}t$$

with

$$\mathscr{L}(\mathscr{P}(y(t)))) = \sum_{i=0}^N \|y_i(t)\|_2^2 + \sum_{i=1}^N \sum_{j \neq i} \|y_i(t) - y_j(t)\|_2^2 .$$

Notice that we have added penalty terms for the controls u_i and multiplers $\lambda_{i,j}$. Furthermore, the stage cost $\mathscr{L}(\mathscr{P}(t))$ is a standard tracking term, and a term to penalize the size of the tube. In addition, the state constraint

$$\mathscr{P}(y(t)) \subseteq \mathbb{X} = \left\{ \xi \in \mathbb{R}^2 \mid \xi_1 \leq 0.35 \right\}$$

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is enforced.

A single-shooting discretization of Problem (23) was implemented in `YALMIP` [15] with `IPOPT` as the underlying optimization solver [16]. The dynamics were discretized using a fixed step Runge-Kutta integrator of order 4 with stepsize $h = 10^{-2}$. A piecewise constant control discretization on 20 equidistant intervals was used.

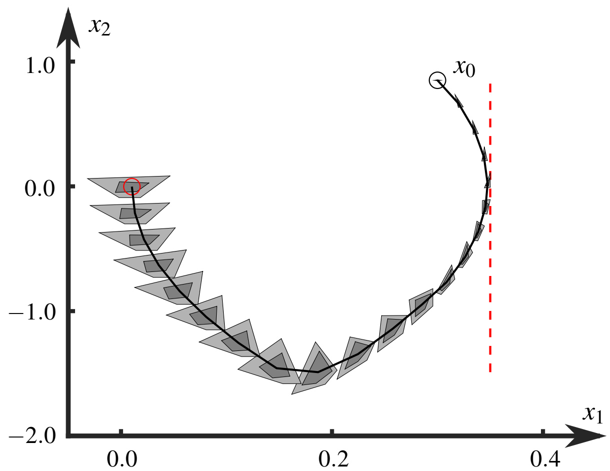

Fig. 1. The optimal inner (dark grey) and outer (light grey) polytopic approximations ($N = 4$ vertices) of a tight RFIT. The trajectories of the vertices start from $y_1(0) = \ldots = y_N(0) = x_0$. The black solid line depicts the trajectory of tube centroid. The red dotted line shows the state constraint $\mathbb{X} = \{y \mid y_1 \leq 0.35\}$.

Figure 1 shows inner (dark grey) and outer (light grey) polytopic approximations (with $N = 4$ vertices), of a tight RFIT of the system. In order to ensure robust constraint satisfaction, the controller rotates the tube sharply around $t = 1.6\,$s. At this point, the state constraint is active. At the end, the controller is able to steer the centroid—i.e., the average of the vertices—of the polytopic tube close to the origin at $T = 8$. The inner approximation, shown in dark grey, as expected, is contained inside the outer one.

VII. Conclusions

A tube MPC technique has been proposed for linear systems with time-varying disturbances. This technique relies on the polytopic parameterization of min-max differential inequality, which provides sufficient conditions for a time-varying convex set-valued function to be an RFIT. The conservatism of the tube was tested via the max-min differential inequality. The stability analysis for the proposed scheme is included in the list of our future work.

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